ECE 574
HW 1  Solution

1. \( AO = \sqrt{x^2 + h_1^2} \)

\( OB = \sqrt{(d-x)^2 + h_2^2} \)

\[ OPL = n_1 \sqrt{x^2 + h_1^2} - n_2 \sqrt{(d-x)^2 + h_2^2} \]

\[ \frac{d}{dx} OPL = \frac{n_1 x}{2 \sqrt{x^2 + h_1^2}} + \frac{n_2 (d-x)}{2 \sqrt{(d-x)^2 + h_2^2}} \]

For \( \frac{d}{dx} OPL = 0 \), then

\[ \frac{X}{\sqrt{x^2 + h_1^2}} = \frac{(d-x)}{\sqrt{(d-x)^2 + h_2^2}} \]

\[ \sin \theta_c = \sin \theta_r \Rightarrow \theta_c = \theta_r \]
Refraction

\[ AO = \sqrt{x^2 + h_1^2} \]

\[ OB = \sqrt{(d-x)^2 + h_2^2} \]

\[ OPL = n_1 \sqrt{x^2 + h_1^2} + n_2 \sqrt{(d-x)^2 + h_2^2} \]

\[ \frac{d}{dx} OPL = n_1 \frac{2x}{2 \sqrt{x^2 + h_1^2}} + n_2 \frac{2(d-x)}{2 \sqrt{(d-x)^2 + h_2^2}} \]

Let \( \frac{d}{dx} OPL = 0 \), then

\[ \frac{x}{\sqrt{x^2 + h_1^2}} = \frac{(d-x)}{\sqrt{(d-x)^2 + h_2^2}} \]

\[ \sin \theta_i \]

\[ \sin \theta_+ \]

\[ \Rightarrow n_1 \sin \theta_i = n_2 \sin \theta_+ \]
2. Numerical Problem: Refraction

<table>
<thead>
<tr>
<th></th>
<th>$\lambda = 10 \mu m$</th>
<th>$\lambda = 1.6 \mu m$</th>
<th>$\lambda = 1.55 \mu m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GaAs</td>
<td>3.3&lt;sup&gt;1&lt;/sup&gt;</td>
<td>3.30&lt;sup&gt;2&lt;/sup&gt;</td>
<td>3.16&lt;sup&gt;3&lt;/sup&gt;</td>
</tr>
<tr>
<td>Air</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Al&lt;sub&gt;0.5&lt;/sub&gt;As&lt;sub&gt;0.5&lt;/sub&gt;Sb</td>
<td>3.30&lt;sup&gt;2&lt;/sup&gt;</td>
<td>3.16&lt;sup&gt;3&lt;/sup&gt;</td>
<td></td>
</tr>
<tr>
<td>InP</td>
<td>1.46</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td>Fiber Core</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fiber Clad</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


Angle of Refraction: Your code should output a plot with the angle of incidence on the x-axis and the angle of refraction on the y-axis – similar to those below.

Critical Angles: Your code should produce answers similar to the ones below

<table>
<thead>
<tr>
<th>Material Pair</th>
<th>Angle of Refraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>GaAs-Air</td>
<td>~17 degrees</td>
</tr>
<tr>
<td>AlAsSb-InP</td>
<td>~73 degrees</td>
</tr>
<tr>
<td>Fiber Core-Clad</td>
<td>80.5 degrees</td>
</tr>
</tbody>
</table>

Part A: GaAs-Air

Part B: AlAsSb-InP

Part C: Fiber Core-Clad
3. Numerical Problem: Diffraction

For each point on the screen, \( y_i \), calculate the electric field amplitude, \( E_i \). This can be done numerically by calculating the pathlength difference, \( \Delta x_i \), between the individual point sources, \( a_i \), along the aperture. The phase of the pathlength difference can be determined by multiplying it by the wave number, \( k = 2\pi/\lambda \), of the incident light. Due to interference, each point source will contribute a different amount of phase to the total electric field at each point along the screen.

**Pathlength Difference**

\[
\Delta x_i = (a_1 - a_i)\sin(\theta_i), \quad \theta_i = \sin^{-1}(y_i/d)
\]

**Phase Difference**

\[
\Delta \varphi_i = k\Delta x_i = (2\pi/\lambda)\Delta x_i
\]

**Wave Interference**

Consider two waves of equal amplitude and frequency, but with a phase difference. When these waves interfere (i.e. add), how can one calculate the resulting wave?

\[
E_1 = E_0\cos(\omega t - kx) \quad E_2 = E_0\cos(\omega t - kx + \Delta \varphi)
\]

\[
E_1 + E_2 = 2E_0\cos(\omega t - kx)\cos(\Delta \varphi/2)
\]

Wave interference of identical sources is explained nicely in Knight’s physics text (21.5 - 21.6).

To calculate a normalized intensity, only the component contributed by the phase difference needs to be considered. The total electric field at each point on the screen is simply the sum of all the fields originating from each point source – fields which differ from one another only by a phase shift as described above.

**Normalized Intensity**

The intensity is proportional to the electric field amplitude squared. The intensity can be normalized by dividing by the maximum intensity (this should be at the center of the screen).

\[
I_i = \frac{E_i^2}{I_{\text{max}}}
\]

a) You can calculate the expected minima with the following formula:

\[
\sin(\theta_m) = m\lambda/a \quad y_m = L\tan(\theta_m) \quad m = \text{any non-zero integer, or the orders of the minima}
\]

<table>
<thead>
<tr>
<th>( m )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_m ) (cm)</td>
<td>-34.02</td>
<td>-17.32</td>
<td>-7.75</td>
<td>7.75</td>
<td>17.32</td>
<td>34.02</td>
</tr>
</tbody>
</table>

These minima are plotted in red below. With enough point sources, you should find a good match.
b) In the code provided, the Fraunhoffer formula is used to analytically compute the diffraction pattern, which is merely used as a comparison to the numerical model described above. Showing the Fraunhoffer results is not a requirement of the assignment. Keep in mind, if you only used the Fraunhoffer formula and did not use a numerical model, then you cannot answer this question. The Fraunhoffer formula is the result of using an infinite number of point sources (i.e. a line source).

By starting with N=1, the model will generate a single line of uniform amplitude, which is clearly non-physical. With N = 2, the first pair of minima will be correctly calculated. As more point sources are added, the higher order minima can be more accurately approximated. I found that between N=20 and N = 50 was enough to approximate the second order minima. Whichever N you chose will be considered correct if your data supports your conclusion.
c) With a wavelength of 20 µm, most of the light is transmitted in a near uniform pattern, with maximum brightness in the center of the screen. The bright intensity at the center is consistent with ray optics, but the region of uniform intensity extends to infinity, which is clearly a non-physical prediction. Keep in mind that we assumed paraxial rays when determining the phase shift, so if the intensity fails to approach zero at large angles, the approximation breaks down.

\[ \lambda = 20 \mu m \]

You should observe that the longer wavelength has an intensity pattern with a broader central maximum and fewer orders of minima. The shorter wavelength has a narrower central maximum with more orders of minima. Furthermore, the one half reduction in wavelength results in twice as many minima (if you count infinity).
With longer wavelengths, the slit now has sub-wavelength dimensions, which results in an intensity pattern without any fringes. Without identifying any minima in the intensity pattern, one cannot determine the slit size. To easily determine the size of the slit from the interference pattern, light with a shorter wavelength must be used (in fact, light with a considerably shorter wavelength will work, so long as the Fresnel number is much less than one).